Numerals are doubly bounded: evidence from exclusives and polarity*

Lucas Fagen

The University of Chicago

1. Introduction

A central question in number semantics concerns the source of upper-bounding inferences in sentences with numerals like (1).

(1) Three cats won awards. $\rightarrow$ No more than three cats won awards.

According to the classic analysis of numerals going back to Horn (1972), numeral sentences assert existential, lower-bounded truth-conditions (2). The bare assertion in (2a) is "one-sided" because the existence of three award-winning cats does not in itself preclude larger numbers of cats from having won awards. The inference that no more than three cats won awards is treated as a scalar implicature, derived either via pragmatic enrichment or compositional exhaustification (2b). Recent proponents of this view include Spector (2013) and Bylinina and Nouwen (2018).

(2) a. $\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = 3]$
b. $\neg\exists y[\mathbf{cats}(y) \wedge \mathbf{won}(awards)(y) \wedge \#(y) > 3]$

Another approach treats numerals as generalized quantifiers over degrees that impose upper bounds directly by maximizing over the set of degrees that satisfy a derived degree property. (3) conveys "two-sided" truth-conditions: if fewer than three cats won awards, then the number 3 would not satisfy the property of being a number of award-winning cats; if more than three cats won awards, then 3 would not be the maximum. Recent proponents of this view include Kennedy (2015) and Buccola and Spector (2016).

(3) $max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = n]] = 3$

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Both approaches ultimately converge on the same meaning, but take different paths to get there. For an overview of the various arguments for and against each approach, see Bylinina and Nouwen (2020).

This paper makes a novel argument for the degree-maximizing approach based on the interactions of numerals, exclusive modifiers, and negative polarity items (NPIs) in English. The paper aims to explain why the English exclusives *just*, *merely*, and *mere* can license weak NPIs like *any* and *ever* when they compose with numerals and other degree terms. Unlike their lexical cousin *only*, these particular exclusives generally do not license NPIs. VP-modifying *just* and *merely* fail to license NPIs in the VP (4); similarly, NPIs are unacceptable in the scope of NP-modifying *just*, *merely*, and *mere* (5).

(4) a. My cat Gertrude only ever eats [kibble]$_F$.
b. #My cat Gertrude just ever eats [kibble]$_F$.
c. #My cat Gertrude merely ever eats [kibble]$_F$.

(5) a. Only [Mikaela Shiffrin]$_F$ has ever won 15 races in the same calendar year.
b. #Just [Mikaela Shiffrin]$_F$ has ever won 15 races in the same calendar year.
c. #Merely [a child]$_F$ won any awards.
d. #The mere [child]$_F$ who won any awards was a prodigy.

When *just*, *merely*, and *mere* compose with degree terms, however, NPIs are acceptable, as the naturally occurring data in (6) shows (examples from COCA).

(6) a. Just two were ever made.
b. Just six American skiers have ever won two medals in a single Olympics.
c. From Friday to Wednesday, over the course of twelve screenings, merely 95 seats ever sat empty.
d. A mere 750 were ever subsequently convicted of any offense.
e. The Librarian was one of just three Lifeworkers ever honored with that rank.
f. Bowen recalls how her own rarefied status – as one of just six women ever to be elected to statewide office – was brought home at her 2007 swearing-in, when she was surrounded by past and present statewide elected officers.
g. It was the only Colorado site ever to be fully cleaned – just one of 68 ever completed in America.
h. Scientists have managed to wipe out just one human pathogen ever.

To my knowledge, this was first observed by Breheny (2008). Coppock and Beaver (2013) discuss the pattern with *mere*, analyzing data like (7) as a scope contrast.

(7) a. A mere three people gave me any feedback.
b. Of all these children and teens struggling with emotional and behavioral problems, a mere 30% receive any sort of intervention or treatment.
c. A mere 4% listed it at all.

d. At present a mere minority of the Chinese overseas have any living memory of the ancestral land.
e. #A mere smile from him would make any difference.

Mere in (7e) takes NP-internal scope, while the generalized quantifiers modified by *mere* in the other examples take sentential scope. Since NPIs must appear in the scope of their licensors, Coppock and Beaver predict the contrast between (7e) and the other examples. This is an incomplete account of the data. It's no coincidence that all of their examples in (7) involve degree terms: not just any quantifier will do. In addition to numerals, *just*, *merely*, and *mere* can license NPIs when they compose with degree-denoting *a few*, *a minority*, and *a fraction*, but not *some*, singular indefinites, or indefinite bare plurals (8).

(8) a. Just/merely/a mere [two]$_F$ were ever made.
b. Just a [few]$_F$/merely a [few]$_F$/a mere [few]$_F$ American skiers have ever won two medals in a single Olympics.
c. Just a [minority]$_F$/merely a [minority]$_F$/a mere [minority]$_F$ of the seats ever sat empty.
d. Just a [fraction]$_F$/merely a [fraction]$_F$/a mere [fraction]$_F$ were ever subsequently convicted of any offense.
e. #Just/merely [some]$_F$ were ever made.
f. #Just a [child]$_F$/merely a [child]$_F$/a mere [child]$_F$ received any sort of intervention or treatment.
g. #Just/merely/mere [American]$_F$ skiers have ever won two medals in a single Olympics.

A final puzzle to be explained is that in order to license NPIs, *just/merely/mere* and degree terms need to compose directly; cooccurrence is not enough. NPIs are licensed in (9a) with intonational focus on the numeral, but not when other expressions are focused (9b) or when the exclusives compose with a larger expression containing the numeral (9c).

(9) a. I brought just/merely/a mere [three]$_F$ pencils to any of the exams.
b. #I brought just/merely/a mere three [pencils]$_F$ to any of the exams.
c. #I just/merely brought [three]$_F$ pencils to any of the exams.

This data provides evidence for two theoretically significant conclusions: one about how exclusives vary, and one about how numeral upper bounds are derived. In the realm of exclusive semantics, *just*, *merely*, and *mere* need to be distinguished from *only* in their polarity profiles. Section 2 discusses a way to do this by restricting the orderings exclusives impose on alternatives.

In the realm of degree semantics, numerals and other degree terms need to be distinguished from other quantifiers in some way that explains the polarity pattern. Section 3 argues that this data supports a degree-maximizing analysis of numeral upper bounds, which conspire with exclusives to create an NPI environment. Bare numeral sentences do not license NPIs because both bounds are at-issue. But exclusives presuppose lower bounds, so only the upper bound is at-issue in a sentence with exclusives and numerals. In such a context, maximizing over degrees reverses logical strength. Key to the analysis is that the relevant upper bound comes from the maximality function itself, not the exclusive. Section 4 discusses a challenge from *zero*.

2. *Just ever*

Why does *only* license NPIs, but not *just*, *merely*, or *mere*? The pattern with the VP-modifiers is repeated in (10a). The contrast holds across clause boundaries (10b), and applies to *any* as well as *ever* (10c).

(10) a. Gertrude only/#just/#merely ever eats [kibble]$_F$.
b. I only/#just/#merely thought that Gertrude ever ate [kibble]$_F$.
c. I only/#just/#merely brought [Gertrude]$_F$ to any of the cat shows.

Coppock and Beaver (2014), building on Beaver and Clark (2008), propose a unified lexical entry schema for exclusives as operating on an ordered scale of alternatives $\geq_s$, modeled as answers to the current Question under Discussion (CQ) (Roberts (1996/2012)). Exclusives presuppose a lower-bounding MIN statement that a true answer to the CQ is ranked at least as high as the prejacent, and assert an upper-bounding MAX statement that no true answer is ranked higher (11). Variation between exclusives is captured by parameterizing them for semantic type: adjectival exclusives and VP-modifiers are analyzed as property modifiers instantiating "P-ONLY" (11d), while NP-modifiers are analyzed as modifiers of generalized quantifiers instantiating "Q-ONLY" (11e). Each type can be generated by applying the Geach rule, which maps a function of type $\langle a,b\rangle$ to a function of type $\langle\langle c,a\rangle,\langle c,b\rangle\rangle$, to the $\langle p,p\rangle$ propositional modifier ONLY (11c). So to generate P-ONLY, $c = e$; to generate Q-ONLY, $c = \langle e,p\rangle$.

(11) a. $\text{MIN}_s(p) = \lambda w.\exists q_{\in \text{CQ}_s}[q(w) \wedge q \geq_s p]$
b. $\text{MAX}_s(p) = \lambda w.\forall q_{\in \text{CQ}_s}[q(w) \rightarrow p \geq_s q]$
c. $[\![\text{ONLY}_s]\!] = \lambda p \lambda w : \text{MIN}_s(p)(w).\text{MAX}_s(p)(w)$
d. $[\![\text{P-ONLY}_s]\!] = \lambda P_{\langle e,p\rangle} \lambda x_e.\text{ONLY}_s(P(x))$
e. $[\![\text{Q-ONLY}_s]\!] = \lambda Q_{\langle\langle e,p\rangle,p\rangle} \lambda P_{\langle e,p\rangle}.\text{ONLY}_s(Q(P))$

Exclusives also vary by scale structure, alternating between "complement exclusion" readings that are interpreted exhaustively (12a), and "rank-order" readings that are not (12b).

(12) a. Gertrude only/just eats caviar. → Gertrude eats nothing **other** than caviar.
b. Frederick is only/just a kitten. → Frederick is nothing **higher** than a kitten.

Complement exclusion readings like (12a) order the alternatives as a boolean lattice in which the higher alternatives entail the lower ones. Rank-order readings like (12b) lack boolean structure. Complement exclusion sentences specify the ordering $\geq_s$ as entailment, while rank-order alternatives can be ordered by other relations. The ordering relation is a distinct parameter from scale structure, since scales that lack boolean structure can still be ordered by entailment when the atomic alternatives entail each other (e.g., Horn scales). Lattice scales can be ordered by other relations too.

In Coppock and Beaver's system, $\geq_s$ is provided by context, but exclusives have "violable preferences" for different orderings: *only* "prefers" entailment scales and *just* rank-order scales. It's not clear exactly how violable these preferences are. Both *only* and *just* superficially allow both readings in (12). Horn (2000), pointing to their differential projection behavior in data like (13), analyzes *only* and *just* as absolutely restricted to entailment vs. rank-order readings, respectively. The prejacent projects under negation when the alternatives are ordered by entailment because the higher alternatives entail the lower ones. When the alternatives are logically independent or mutually exclusive, as on a rank-order scale, it does not. Horn argues that projection of the prejacent in the *only* but not the *just* examples in (13) indicates that *only* excludes by entailment and *just* by rank.

(13) a. They're not just/#only engaged, they're married.
 b. I didn't just/#only get a B, I got an A.

There is some speaker variation re the acceptability of rank-order readings with *only*, though, as Coppock and Beaver observe[1]. Moreover, projection of the prejacent can only be taken as a test of whether the alternatives asymmetrically entail each other, not whether the ordering is specified as entailment. Even the prejacent of *merely*, which Coppock and Beaver analyze as requiring an "evaluative" scale whose alternatives are ranked according to what the speaker considers good or bad, can project when the higher-ranked alternatives entail it (14).

(14) a. He did not merely win the Masters by an eye-popping 12 strokes in April 1997. He also appears in frequent advertisements as a high-profile endorser. (COCA)
 b. As far as cello and violin making goes, Stradivari did not merely use Newton's methods, he exhausted them. (COCA)
 c. Food, house and clothing are mine forever. Therefore not merely do effort and labour cease, but also hatred and bitterness. (Virginia Woolf, *A Room of One's Own*)

So it's not so straightforward to recover the precise ordering relation from what the alternatives are. In Fagen (2022), however, I argue that Horn was right to restrict *just* to rank-order scales, and that the polarity contrast between *only* vs. *just* and *merely* can be explained if *only* can order alternatives by entailment but the other exclusives cannot. Assuming Chier-

[1] Horn claims to accept them only if interpreted "echoically or metalinguistically" (Horn (2000): pg. 149). Coppock and Beaver provide several corpus examples, including (i).

(i) So it looks like these two aren't only engaged, but they may have already had a dreamy miliblonde wedding.

chia's (2013) analysis of NPIs as existential quantifiers with maximally wide domains (15a) that trigger exhaustification over domain alternatives (15b), NPIs are grammatical when exhaustification does not contradict the prejacent (16). In the downward entailing (16a), the exhaustification conjunct says that there is no wider domain of time in which Gertrude doesn't eat kibble, which is entailed by the prejacent. In the upward entailing (16b), the exhaustification conjunct says that there is no narrower domain of time in which Gertrude eats kibble, which contradicts the prejacent. By treating *exh* as scalar, sensitive to the same orderings that exclusives are, we can derive the polarity data.

(15) a. $[\![\text{ever}]\!] = \lambda e.\exists i_{\subset \textbf{ever}}[\tau(e) = i]$
b. $[\![exh]\!] = \lambda p \lambda w.p(w) \wedge \forall q_{\in \text{ALT}(p)}[{}^a q(w) \rightarrow {}^a p \geq_s {}^a q]$

(16) a. $[\![exh[\text{Gertrude doesn't ever eat kibble}]]\!] = \lambda w.\neg\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i](w) \wedge \forall q_{\in \text{ALT}(p)}[q(w) \rightarrow (\neg\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i] \rightarrow q)]$
b. $[\![exh[\text{Gertrude ever eats kibble}]]\!] = \lambda w.\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i](w) \wedge \forall q_{\in \text{ALT}(p)}[q(w) \rightarrow (\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i] \rightarrow q)]$

The key innovation in Fagen (2022) is to model sentences with exclusives and NPIs as drawing on a single alternative set, represented as the union of the exclusive's focus alternatives (F-ALTs) and the NPI's domain alternatives (D-ALTs), all ordered by the same relation. If that relation were rank rather than entailment, MAX would not reverse the ordering: an NPI would still be ranked lower than its domain alternatives despite their negation being entailed.

The input to *exh* in an exclusive sentence is the MAX statement, where the alternatives consist of equivalent MAX statements with narrower subdomains. These are entailed by the prejacent. When $\geq_s$ is entailment, they are also therefore ranked lower, and *exh* excludes nothing. When $\geq_s$ is rank, they are still ranked higher, and *exh* excludes them (17b). (17a) says that Gertrude eats nothing ranked higher than kibble in the widest domain, but there is no narrower domain in which Gertrude eats nothing ranked higher than kibble. This is a contradiction.

(17) a. $\#[\![exh[\text{Gertrude just ever eats kibble}]]\!] = \lambda w : \text{MIN}_s(\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i])(w).\text{MAX}_s(\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i])(w) \wedge \forall q_{\in \text{ALT}(\text{MAX}(p)}[q(w) \rightarrow (\exists i_{\subset \textbf{ever}}[\tau((\textbf{eat})(k)(g)) = i] \geq_s q)]$
b. $\text{ALT}(\text{MAX}_s(p)) = \{\text{MAX}_s(\langle kibble,\ ever\rangle),\ \cancel{\text{MAX}_s(\langle kibble,\ sometimes\rangle)},\ \cancel{\text{MAX}_s(\langle kibble,\ often\rangle)}...\}$

It follows that NPIs can't appear in the scope of rank-order exclusives without contradiction[2]. With this result in hand, section 3 proposes an analysis of how *just*, *merely*, and *mere* interact with degree terms to create an NPI environment.

[2]Except in sentences with other NPI licensors. In (i), negation uses up the D-ALTs before *just* can see them.

(i) There just aren't any $[\text{American}]_F$ skiers in the competition (every other country was represented).

3. *Just three*

Since *just*, *merely*, and *mere* do not license NPIs on their own, the explanation for data like (18) must lie in degree semantics.

(18) a. Just three cats won any awards.
b. #Just some cats won any awards.

A lower-bounded existential numeral semantics (19a) makes no systematic distinction between numerals and other quantifiers, and would not predict the contrast in (18). Treating *three cats* as an existential quantifier, as Coppock and Beaver (2013) do, would derive (19b), where *three cats* composes with Q-ONLY *just*, and the entire proposition ends up in the scope of MIN/MAX.

(19) a. $[\![\text{Three cats won awards}]\!] = \exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = 3]$
b. $[\![\text{Just three cats won any awards}]\!]$
$= \lambda P.\text{ONLY}_s(\exists x[\mathbf{cats}(x) \wedge \#(x) = 3 \wedge P(x)])(\lambda x.\mathbf{won}(awards)(x))$
$= \text{ONLY}_s(\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = 3])$

This won't do: *any* is in the scope of *just*. Equivalent truth-conditions are exactly what rule out NPIs in (18b), in which Q-ONLY *just* modifies an existential quantifier that isn't a degree term.

Instead, the relevant upper bound must come not from MIN/MAX but the numeral itself. To predict (18), we need a numeral semantics that maximizes over degrees (20).

(20) $[\![\text{Three cats won awards}]\!] = max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = n]] = 3$

These "two-sided" truth-conditions are not monotonic, and would not be predicted to license NPIs on their own[3]. (Compare the upward entailing (19a)). Since both bounds are at-issue, there is no entailment relation in either direction between sets and subsets. Exactly three cats can win awards without exactly three cats winning gold medals; exactly three cats can win gold medals without exactly three cats winning awards. Composing in the right way with exclusives, which presuppose lower bounds, will render the scope of the *max* function downward entailing.

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[3] Aside from examples like (i), from Linebarger (1987).

(i) #(Exactly) four people in the whole world have ever read that dissertation: Bill, Mary, Tom, and Ed.

See Crnič (2014) for discussion of such cases, which are especially puzzling for a two-sided numeral semantics. Sauerland (2013) proposes an analysis of *exactly* as a two-sided exclusive that asserts both the prejacent and the negation of focus alternatives. In contexts where the prejacent is already presupposed, this analysis would predict *exactly* to license NPIs for the same reason that *only* does.

Kennedy (2015) proposes (21) as the lexical meaning of numerals. On this account, numerals are degree quantifiers that take scope over derived degree properties.

(21) $[\![three]\!] = \lambda D_{\langle d,p\rangle}.max[n|D(n)] = 3$

Suppose exclusives had a type $\langle\langle\langle d,p\rangle,p\rangle,\langle\langle d,p\rangle,p\rangle\rangle$ entry like (22a), as modifiers of degree quantifiers (similar to Q-ONLY, but defined on degrees rather than entities, derived by Geaching with $c = \langle d,p\rangle$). Then "DQ-ONLY" would compose with *three* to produce (22b), with the entire proposition once again in the scope of MIN/MAX.

(22) a. $[\![\text{DQ-ONLY}_s]\!] = \lambda Q_{\langle\langle d,p\rangle,p\rangle}\lambda D_{\langle d,p\rangle}.\text{ONLY}_s(Q(D))$
b. $[\![$Just three cats won any awards$]\!]$
$= [\![\text{DQ-ONLY}]\!](\lambda D_{\langle d,p\rangle}.max[n|D(n)] = 3)(\lambda n.\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = n])$
$= \text{ONLY}_s(max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = n]] = 3)$

These truth-conditions are equivalent to (19b), with one difference: on this view, numeral alternatives do not stand in an entailment relation. These "two-sided" statements about cardinality are mutually exclusive alternatives ordered by rank. But *any* is still in the scope of *just*, so this doesn't work either.

The problem with formulas like (19b) and (22b) is that MIN/MAX scopes over the entire proposition. As discussed in section 2, NPIs can't occur in the scope of rank-order MIN/MAX without an independent NPI licensor. There is no independent licensor in sentences like (18a). Contra Coppock and Beaver, then, the NPI must be outside the MIN/MAX environment. To derive an NPI-friendly environment, we need to confine the scope of *just* to the numeral.

Specifically, let's follow Buccola and Spector (2016)'s "SMax" analysis of numeral maximality (short for syntactic maximality), in which numerals lexically denote type *d* expressions (23a), but can be shifted to degree quantifier denotations by a maximality operator `isMax` (23c)[4]. I'll assume type *d* entries for expressions like *a few* and *a minority* too. Numerals saturate a degree argument position introduced by a Bresnan (1973)-style `many` (23b), which takes a degree as input and returns a property that composes intersectively. `isMax`, a descendant of Heim (2006)'s π operator, maps a degree to a generalized quantifier over degrees equivalent to Kennedy's (21). The metalanguage *max* function is defined in (23d)[5].

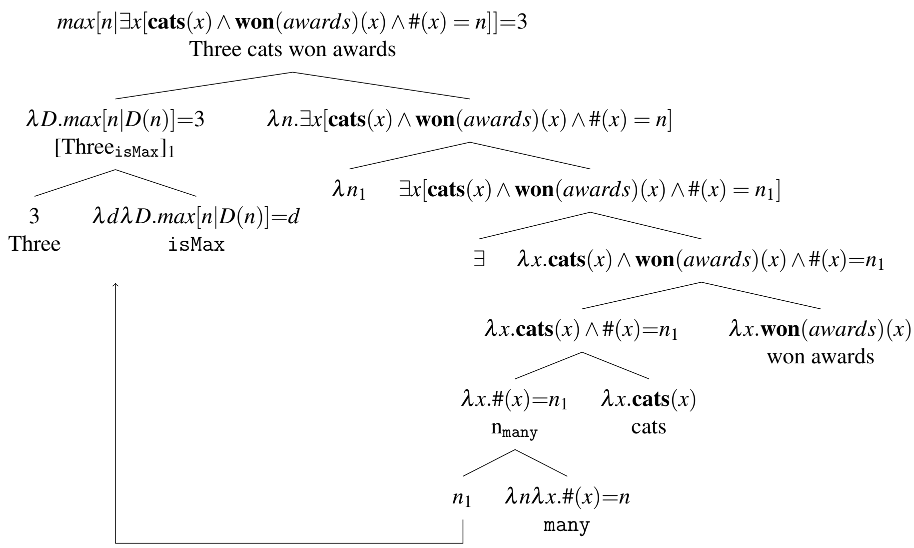
[4]Technically, degree quantifier entries could still work here as long as `isMax` is also available. As Kennedy (2015) shows, degree quantifiers like (21) can be shifted into singular degree terms via successive application of Partee (1986)'s BE and IOTA.

[5]Buccola and Spector (2016) compare several different implementations of numeral maximality. What matters for this paper is to decompose it from the lexical meanings of bare numerals. Either "standard" or "logical" maximality would work here. Buccola and Spector's definition of the "standard" maximality function included an *else* condition specifying that $max(\varnothing) = 0$. I've omitted this condition for reasons discussed in section 4. I've also said nothing about how modified numerals implement maximality.

(23) a. $[\![\text{three}]\!] = 3$
b. $[\![\texttt{many}]\!] = \lambda d_d \lambda x_e.\#(x) = d$
c. $[\![\texttt{isMax}]\!] = \lambda d_d \lambda D_{\langle d,p\rangle}.max[n|D(n)] = d$
d. $max(D_{\langle d,p\rangle}) = \iota n[D(n) \wedge \forall m[D(m) \rightarrow m \leq n]]$

For a bare numeral sentence, Buccola and Spector's analysis would derive (24), in which *three* composes with `isMax` to create a type $\langle\langle d,p\rangle,p\rangle$ degree quantifier $three_{\texttt{isMax}}$ that takes scope out of `many`'s type *d* argument position and maximizes over the derived degree property of being a number of award-winning cats.

(24)



These are exactly the same truth-conditions proposed by Kennedy, but *three* now saturates one of `isMax`'s argument positions and can in principle scope out again. This feature of the analysis makes possible an account of the polarity data with *just*/*merely*/*mere*. *Three* composes with *just* first. The output is then shifted to a degree quantifier meaning and takes scope twice, once with `isMax` as in (24), and again on its own, leaving `isMax` behind. The result will place NPIs in a downward monotonic environment while confining the scope of MIN/MAX to the degree term.

The exclusive entry we need is a type $\langle\langle d,p\rangle,\langle d,p\rangle\rangle$ modifier of degree properties, which I'll call D-ONLY (25). This can be generated from Coppock and Beaver's ONLY via Geaching with *c* = *d*. This is a new addition to the "exclusive muddle", but it's not such a stretch given a model that already includes type *d* objects.

(25) $[\![\text{D-ONLY}_s]\!] = \lambda D_{\langle d,p\rangle} \lambda n_d.\text{ONLY}_s(D(n))$

A degree property meaning for numerals can be derived by successive application of Partee (1986)'s LIFT and BE operations (26).

(26) a. $\text{LIFT} = \lambda x_\tau \lambda P_{\langle \tau,p\rangle}.P(x)$

b. $\text{BE} = \lambda Q_{\langle\langle \tau,p\rangle,p\rangle}\lambda x_\tau.Q(\lambda y.y = x)$

LIFT maps a singular term to a generalized quantifier over properties that are true of that term. Application of LIFT to *three* returns a set of degree properties true of 3 (27a). BE maps a generalized quantifier to a property true of terms that satisfy the properties in the denotation of the quantifier. Application of BE to LIFT(3) returns the characteristic function of 3 (27b). This is now something that can compose with D-ONLY (27c).

(27) a. $\text{LIFT}(3) = \lambda P_{\langle d,p\rangle}.P(3)$

b. $\text{BE}(\text{LIFT}(3)) = \text{BE}(\lambda P_{\langle d,p\rangle}.P(3)) = \lambda n_d.(\lambda P_{\langle d,p\rangle}.P(3))(\lambda d.d = n)$
$= \lambda n_d.n = 3$

c. $[\![\text{D-ONLY}_s]\!](\text{BE}(\text{LIFT}(3))) = \lambda n_d.\text{ONLY}_s(n = 3)$

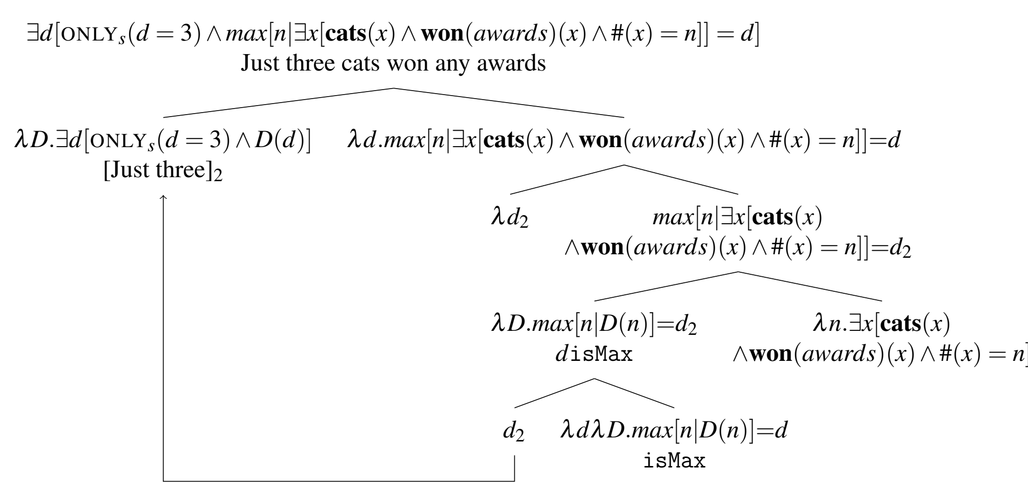
A degree quantifier meaning can then be derived by applying Partee's A, which maps a property to an existential generalized quantifier (28a). Application of A returns a set of degree properties true of a degree presupposed to equal at least 3 and asserted to equal at most 3 (28b)[6].

(28) a. $\text{A} = \lambda Q_{\langle \tau,p\rangle}\lambda P_{\langle \tau,p\rangle}.\exists x_\tau[Q(x) \wedge P(x)]$

b. $\text{A}([\![\text{D-ONLY}_s]\!](\text{BE}(\text{LIFT}(3)))) = \lambda D_{\langle d,p\rangle}.\exists d[\text{ONLY}_s(d = 3) \wedge D(d)]$

This is now something that can take scope over a degree property, deriving (29).

(29)



(29) says that the maximal number of award-winning cats is equal to a number presupposed to equal at least three and asserted to equal at most three. These are the same truth-

[6]This series of type-shifts was inspired by Coppock and Beaver (2014)'s analysis of "minimal sufficiency" sentences like (i). They analyze *just* in (i) as P-ONLY with NP-internal scope. If the current proposal is on the right track, the scope of *just* in (29) is confined to *three* in much the same way.

(i) Just the thought of him sends shivers down my spine.

conditions as before, but the NPI has been rescued from the MIN/MAX environment. The ordering on the D-ALTs can default to entailment now, as desired. The at-issue component of (29) is downward monotonic: if the maximal number of award-winning cats is at most three, then the maximal number of cats who won gold medals is at most three, and so on for all the subdomains. The narrower D-ALTs are therefore entailed by the prejacent, and since they are also ordered by entailment, there are no stronger alternatives for *exh* to exclude. So (30) is not a contradiction, and NPIs are correctly predicted to be licensed.

(30) $[\![exh[\text{Just three cats won any awards}] = \lambda w.(\exists d[\text{ONLY}_s(d=3) \wedge max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x)=n]] = d]])(w) \wedge \forall q_{\in\text{D-ALT}(p)}[q(w) \to (\exists d[\text{MAX}_s(d=3) \wedge max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x)=n]] = d]] \to q)]$

What exclusives contribute should be clear. Since they presuppose lower bounds, the at-issue component of (29) is upper-bounding and downward entailing, unlike non-monotonic bare numeral sentences. The contribution of numeral maximality may be less obvious. Without `isMax`, an equivalent formula with *any* outside the scope of MIN/MAX could still be derived given D-ONLY and Partee's type-shifting rules, as in (31). But this runs into van Benthem's problem (van Benthem (1986)).

(31) $\exists d[\text{ONLY}_s(d=3) \wedge \exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x)=d]]$

(31) says that a not necessarily maximal number of award-winning cats is equal to a number presupposed to equal at least three and asserted to equal at most three. These are much weaker truth-conditions: three or more cats won awards. The at-issue component of (31) is not downward entailing. So NPIs would not be predicted given (31).

A final puzzle this paper set out to solve is why exclusives need to compose directly with degree terms to license NPIs as in (9), repeated in (32). The current proposal predicts this data by requiring *just* and the numeral to scope together: *just* must denote D-ONLY to take degree-internal scope. The numeral must be focused because it's the only expression in the MIN/MAX environment. With NP-modifying Q-ONLY (32b) or VP-modifying P-ONLY (32c), NPIs wind up in the scope of MIN/MAX. This is avoided in (32a).

(32) a. I brought just/merely/a mere [three]$_F$ pencils to any of the exams.
b. #I brought just/merely/a mere three [pencils]$_F$ to any of the exams.
c. #I just/merely brought [three]$_F$ pencils to any of the exams.

4. *Zero*

Around now I should mention that Bylinina and Nouwen (2018) have used a similar argument from negative polarity to reach the opposite conclusion. They aim to explain why *zero* does not license NPIs, unlike the ostensibly synonymous negative quantifier *no* (33).

(33) a. No cats won any awards.
b. #Zero cats won any awards.

Bylinina and Nouwen use this data to argue that maximization over degrees cannot be available for numeral sentences in English. As they show, *zero*'s distribution matches that of other numerals, suggesting that *zero* is not a quantifier but a degree term. *Zero* sentences are puzzling for a traditional theory of plurals as join semi-lattices that close the set of atoms in the denotation of the corresponding singular noun under the join operation (Link (1983)). The existential truth-conditions in (34) are contradictory because the domains of plural nouns like *cats* are not usually taken to include entities with zero quantity. Yet *zero* sentences are not heard as contradictions.

(34) $\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = 0]$

To get around this, Bylinina and Nouwen argue that *zero* sentences provide evidence for a plural semantics that does include minimal elements, proposing to augment the denotations of plural nouns with a bottom element "$\perp$". This ontological tweak makes (34) a tautology. $\perp$ is included in the denotation of every plural object in the domain, which means (34) is true no matter how many cats won awards. But the desired interpretation can be reached via exhaustification (35).

(35) $[\![exh[\text{Zero cats won awards}]]\!]$
$= \exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = 0] \wedge \neg\exists y[\mathbf{cats}(y) \wedge \mathbf{won}(awards)(y) \wedge \#(y) > 0]$

(35) is now contingent: no numbers of cats greater than zero won awards. This proposal predicts the polarity data in (33). NPIs are not licensed in tautologies, which are both downward and upward entailing. (35) becomes non-trivially downward entailing after applying *exh*. But *exh* would also exclude an NPI's D-ALTs when applied to such an environment, resulting in contradiction. A two-sided, degree-maximizing numeral semantics would also derive correct truth-conditions for *zero* (36), but makes wrong predictions about polarity.

(36) $max[n|\exists x[\mathbf{cats}(x) \wedge \mathbf{won}(awards)(x) \wedge \#(x) = n]] = 0$

As long as the range of *max* includes 0, (36) is non-trivially downward entailing prior to application of *exh*. This is so regardless of whether plural denotations include a bottom element. If we include $\perp$, it follows from zero cats having won awards that zero cats won gold medals: both sets of cats consist of $\{\perp\}$. If we don't include $\perp$ but stipulate that $max(\varnothing) = 0$, the downward inference still follows because both sets are empty. Bylinina and Nouwen conclude that *zero*'s polarity profile supports a one-sided numeral semantics.

If the analysis I've proposed in this paper is on the right track, their conclusion about *zero* cannot extend to the positive numerals. We expect *zero* to license NPIs because it denotes the lowest value in the domain. But as the data with exclusives shows, positive numerals do license NPIs when it's presupposed that they denote the lowest value in the domain. So why are NPIs licensed with one but not the other?

This is only a puzzle on the assumption that sentences with *zero* and the positive numerals derive upper-bounding inferences in the same way. Another option would be to remove

0 from the range of the maximality function. The revised definition for *max* in (37) does this explicitly.

(37) $max(D_{\langle d,p \rangle}) = \iota n[n > 0 \wedge D(n) \wedge \forall m[D(m) \rightarrow m \leq n]]$

In a world with $\perp$, properties whose denotations were previously empty now include 0. The maximality function in (37) has no defined output for such properties. If `isMax` composed with *zero*, the resulting truth-conditions would always be undefined: given (37), no *D* has a *max* of 0. Lower-bounded truth-conditions, as in (34), would still be salvageable via exhaustification. This is enough to account for *zero*'s polarity profile: (36) is undefined, and (34) is not an NPI environment. This move would resolve the tension between *zero* and the positive numerals I've sketched in this section, preserving Bylinina and Nouwen's analysis as it applies to *zero* while retaining maximality as an option for positive numeral expressions.

For the purposes of NPI licensing, *max* does not need to be defined on empty sets to ensure downward monotonicity, since a *just n* sentence would still be Strawson DE (von Fintel (1999)) on those D-ALTs for which a two-sided cardinality statement is defined. If the maximal number of award-winning cats is at most three, and some positive number of cats won gold medals, it still follows that the maximal number of cats who won gold medals is at most three. Since $\{\varnothing\}$ is not in the domain of *max* either, similar reasoning would apply if it turns out that pluralities do not include $\perp$ after all.

There have been some challenges to Bylinina and Nouwen's analysis. Elliott (2019) observes that *zero*, unlike the positive numerals, cannot be modified by an exclusive (*#Only zero cats won awards*). This is surprising if pluralities include $\perp$, and especially so if *zero* sentences involve exhaustification. All else being equal, *exh* and *only* should make the same contribution. See also Kennedy (2023) for an analysis of *zero* that retains a traditional plural ontology.

Whatever the correct analysis of *zero* turns out to be, the polarity data with *just*, *merely*, and *mere* provides strong evidence that positive numerals maximize over degrees.

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Lucas Fagen
lfagen@uchicago.edu